Decisions in Economics and Finance

Coherent modelling of mortality patterns for age specific sub-groups

--Manuscript Draft--

Manuscript Number:	DEAF-D-18-00116R1
Full Title:	Coherent modelling of mortality patterns for age specific sub-groups
Article Type:	S.I. : Mathematical & Statistical Methods for Actuarial Sciences & Finance
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Funding Information:	
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Response to Reviewers:	Dear Editor and Reviewers, we have revised the work according to your suggestions and submited two files with a list of changes. Many thanks The Authors

Coherent modelling of mortality patterns for age specific sub-groups

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Abstract

The recent actuarial literature has shown that mortality patterns and trajectories in closely related populations are similar in some respects and that small differences are unlikely to increase in the long run. The common feeling is that mortality forecasts for individual countries could be improved by taking into account the patterns from a larger group. Starting from this consideration, we apply the Three-way Lee-Carter model to a group of countries, by extending the bilinear LC model to a three-way structure, which incorporates a further component in the decomposition of the log-mortality rates. From a methodological point of view, there are several issues to deal with when focusing on such kind of data. In the presence of a three-way data structure, several choices on the pre-treatment of the data could affect the whole modelling process. This kind of analysis is useful to assess the source of variation in the raw mortality data, before the extraction of the rank-one components by the LC-model. The proposed procedure is used to extract an ad hoc time mortality trend parameter for specific age sub-groups. The results show that the proposed strategy leads to a more coherent description of mortality for age specific sub-groups.

Keywords: ANOVA, Human Mortality Database, Lee-Carter Model, Three-way Data Analysis, Tucker-3.

1. Introduction

The world population has becoming more and more connected due to the improvements in transportation, trade and technology. In recent years, the actuarial literature has shown an increasing interest in the mortality experience of "connected" populations which are characterized by similar socio-economic and environmental conditions and by geographical contiguity. If mortality trends for these populations are forecasted separately, the trends will diverge in the long run, even when observations suggest that they have been converging (Li and Lee (2005)). In some contributions, a global convergence in mortality trends has been reported (Wilson (2001)). Non-divergent forecasts are referred to as *coherent*. Therefore, it seems inappropriate to make mortality forecasts for individual national populations in isolation from one another. Individual forecasts, even when based on similar extrapolative procedures, could lead to growing divergence in the long run, in contrast to the expected and observed trend toward convergence. The need for considering aggregate estimate is supported by the empirical evidence that indicates how aggregate series converges toward a stable trajectory in the long run.
This contribution takes as its starting point, on the one hand, the awareness that life tables, and the sets of underlying age specific mortality rates, need to be coherent in

the long run. This result could be achieved by mitigating the differences between each single country. In this respect, a unique synthesis for countries would achieve better results in removing the local wrinkles.
On the other hand, robust patterns can be still observed for each country and across countries for different Age-groups. This phenomenon drives us to take into account the variability in the mortality data structure also with respect to different sources of variability, i.e. Age-groups, gender, etc. We believe that it is advisable that synthesis across countries should be specific for homogeneous sub-groups in order to attain coherent and robust mortality forecasts.
In this contribution, we aim at improving this aspect by taking into account the dissimilarities in the Age-groups while considering coherent synthesis across countries. In other words, we would like to highlight that an exploratory data analysis, preliminary to modelling data, should be undertaken and could be exploited to address different sources of variability in the mortality data. Thus, on the basis of the empirical evidence, we propose to consider different sub-models clustered with respect to different Age-groups, in order to derive specific mortality trend for homogeneous age sub-groups, but still coherent for countries.
The paper is organized as follows. In Section 2 a brief review of the state of the art is introduced. Section 3 briefly recalls the LC models and deepens and interprets the 3-way LC model. The three-way procedure is discussed through a real case study in Section 5. Section 6 concludes.

2. The state of the art

In the last few years, the actuarial literature has focused on models for detecting common population trends (for example D'Amato et al. (2012), Hatzopoulos and Haberman (2013), Njenga and Sherris (2011), Russolillo et al. (2011), Villegas et al. (2017)). These models tend to build on the foundations of the seminal Lee-Carter model (Lee and Carter (1992)), which has some limitations when many populations need to be modelled simultaneously. Investigating long-run equilibrium relationships might provide valuable information about the factors driving changes in mortality, in particular across ages and across countries. This aspect has contributed to the growth of the interest in studying common cross-country longevity trends. For this reason, we have observed a development of country and age-based longevity risk models (Lazar and Denuit (2009), Li and Lee (2005), Bergeron-Boucher et al. (2018), etc.). In 2005, Li and Lee demonstrated the improvement of the mortality projections for individual countries by taking into account the patterns in a larger group. Using the data downloaded from the Human Mortality Database, they applied the Lee Carter (LC) model to a group of populations allowing each its own age pattern and level of mortality, but imposing shared rates of change by age. The so-called Augmented Common Factor LC method, they proposed, is aimed at modelling and forecasting mortality for a group of populations in a coherent way, taking advantage of commonalities in their historical experience and age patterns, while acknowledging their individual differences in levels, age patterns, and trends. In other words, Li and Lee proposed to model the experience of a single population with reference to another coherent population. However, this model requires a great number of parameters, as the age pattern and the time-index parameters need to be provided for the common factor and for each population's specific deviation. Several other models followed the Li and Lee's idea; the literature refers to these models as *multi-population models* or *coherent mortality models*. These studies suggest dependence across multiple populations and common long run relationships between countries (for instance see Lazar and Denuit (2009)).

More recently, Russolillo et al. (2011) extended the LC model by using a three-way analysis and applied the methodology to ten European countries. The idea which inspired the authors was that if two or more populations are assumed to have similar age by time series patterns of experienced mortality, a multi-way analysis can be carried out to describe these patterns. Multi-way techniques are used for analyzing data presented in multidimensional data arrays. In this framework, the life table for different populations can be organized in a three-way array (each dimension is called a “way”, while the entities in each way are the “modes”), where the three indices fixed for the three *modes* refer to *time*, *age* and *country*. The use of a multi-way model allowed the authors to explore mortality patterns of different time-by-age matrices unfolded according to the different countries. In their analysis, Russolillo et al. (2011) found that similar time patterns of mortality should be taken into account.

Bergeron-Boucher et al. (2018) have extended to a Compositional Data Analysis framework the concept of using a three-way model for forecasting, theorized by Russolillo et al. (2011). They explored the opportunity of simultaneously modeling and forecasting the compositional structure of mortality for different populations. In order to test their methodology, the authors carried out a multi-way analysis of the life tables of Canadian provinces and territories.

In this contribution, we start from the consideration that similar time patterns of mortality should be taken into account (Russolillo et al. (2011)) and we take a step forward by exploring the dataset and running a Hierarchical Cluster analysis to have useful insights on how to aggregate the data. Indeed, in presence of disaggregated data, we propose to arrange a three-way data array and assess the different sources of variability. The aim of this study is firstly to build up a more robust analysis producing different sub-models for age homogenous sub-populations. Then, we aim at obtaining specific mortality projections for age homogenous sub-populations, provided that similar age patterns across populations are also an important requirement for using a three-way analysis to forecast mortality.

Thus, we consider the three-way LC model investigated through a three-way analysis of variance with fixed effects, where each cell is given by the log-mortality rate in a given year of a specific Age-group for each of the concerned countries. In other words, we take into account a three-way array, indexed by the mortality rates for years, age and countries. In presence of these kind of data there are several choices on data pre-treatment that will affect the whole data modelling. The textbook by Kroonenberg (1983), gives a detailed mathematical description of the model and discusses advanced issues such as data preparation/scaling and core rotation. The Tucker3 model with orthogonal factors is also known as a 3-way PCA (Principal Component Analysis). The Tucker3 model allows for extracting different numbers of factors in each of the modes. Generally speaking, the advantage of introducing a third way in the analysis of a traditional ears per age-class mortality data is effective if we can hypothesize that the decomposition along the third way brings information in terms of differences across the levels of the third mode (here Countries). At a first glance, the average levels of mortality can be compared, but the comparison can be extended to different statistics. As we are interested in modelling mortality trends, we will put a certain emphasis on the shape of different trends when comparing countries and take into account the similarities when divided for age homogeneous sub-populations .

3. Recalling the LC models

In this section, the two and three-way Lee-Carter models are recalled. The two-way LC model is defined in Equation (1), by considering the mean centered log-mortality rates $\tilde{y}_{ij}$ measured over two modes: Years *I* and Age-groups *J*:

$$\tilde{y}_{ij} = \ln(y_{ij}) - \alpha_j = \kappa_i \beta_j + \varepsilon_{ij} \qquad i = 1, \cdots, I;\ j = 1, \cdots, J \tag{1}$$

where α_j represents the *j-th* average effect of the age independent of time. The other two terms are κ_i, a time-varying parameter reflecting the trend in the level of mortality, and the element β_j, an age-specific term that represents how mortality at each age varies when the general level of mortality changes. The ε_{ij} is the error term.
The model in (1) can be stated in matrix form highlighting the outer product in the Singular Value Decomposition of the matrix $\tilde{\mathbf{Y}}$:

$$\begin{bmatrix} \tilde{y}_{11} & \cdots & \tilde{y}_{1J} \\ \ddots & \tilde{y}_{ij} & \ddots \\ \tilde{y}_{I1} & \ddots & \tilde{y}_{IJ} \end{bmatrix} = \sqrt{\lambda_1} \begin{bmatrix} v_{i1} \\ \vdots \\ v_{I1} \end{bmatrix} \begin{bmatrix} u_{11} & \cdots & u_{1J} \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} & \cdots & \varepsilon_{1J} \\ \vdots & \varepsilon_{ij} & \vdots \\ \varepsilon_{I1} & \cdots & \varepsilon_{IJ} \end{bmatrix} \tag{2}$$

We highlight that Equation (1) and Equation (2) state a low-rank approximate decomposition of the mean centered log-mortality rates. The rank-one solution in Equation (2), written in terms of the first singular value and the first left and right singular vectors, is commonly used. The error term stands for the degree of approximation, which should show uninformative and non-systematic patterns/trends in the data. Equation (2) is obtained posing $\kappa_i = \sqrt{\lambda_1} v_{i1}$ for $i = 1, \ldots, I$ and $\beta_j = u_{1j}$, for $j = 1, \ldots, J$. Otherwise, further components (which are orthogonal and less informative) could be considered in the additive model. However, due to the strong linear correlations present in the original data, the rank-one approximation usually holds. Equation (2), known as the traditional LC model can be directly related to as Principal Component Analysis of the centered log-mortality rates.
Notice that the residual term has to be analyzed for the presence of systematic patterns that deviate from randomness. For instance, a sphericity test on the singular values or usual empirical rules should be considered (Jackson 1993) to judge the meaningless of the disregarded singular values resulting from the data matrix decomposition stated in Equation (2).

The three-way extension of the LC model has been provided by Russolillo et al. (2011) as an exploratory tool in the framework of the Tucker3 model, which is the natural extension of principal component analysis for three-way data.
For the generic element of the three-way array, we have:

$$\tilde{y}_{ijl} = \ln(y_{ijl}) - \alpha_{jl} = \kappa_i \beta_j \gamma_l + \varepsilon_{ijl} \qquad i = 1, \cdots, I;\ j = 1, \cdots, J;\ l = 1, \ldots, L \tag{3}$$

In Equation (3), we refer to the index *l* to vary across *L* Countries, so that α_{jl} is the age-by-country average death level independent of time, while β_j and κ_i have the same interpretation as in the two-way LC model (Equation 1). The term γ_l represents the term associated to the Country-effect. Specifically, the term $\kappa_i\beta_j\gamma_l$ is specified as an outer products conformed to reconstruct the three-way array, while the error term represents the gap between actual data and the approximation provided by the reconstruction formula because a less than full rank solution is considered. We recall that the first component of the three-way singular value decomposition is usually enough for this kind of application. The consideration of a second orthogonal component is sometimes useful for the sake of graphical representation or in the case that the one-rank approximation shows unsatisfactory results in terms of explained variance.

A considerable advantage of the Tucker 3 model is that it is possible to choose a different number of components for each separate way, as is evident from its decomposition formula in matrix form and looking at the role played by the so-called core array:

$$\tilde{y}_{ijl} \approx \sum_{p=1}^{d_1}\sum_{q=1}^{d_2}\sum_{r=1}^{d_3} g_{pqr}k_{ip}b_{jq}c_{lr} \tag{4}$$

In Equation (4), the approximation of the ijl-th element of $\tilde{\mathbf{Y}}$ is a sum of the possible $d_1d_2d_3$ outer products, each weighted by the corresponding elements of the core array $\mathbf{G}$, having dimension (d_1, d_2, d_3). Usually the component matrices are column-wise orthogonal. For our purposes, we set $d_1 = d_2 = d_3 = 2$, so that useful graphical representation can be provided. The array representation is stated in the following Figure 1:

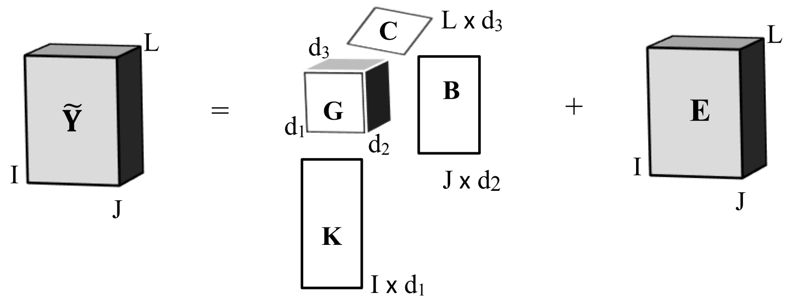

Figure 1. The Tucker3 model as weighted sum of outer products between the components stored as columns in K, B and C. The core array G allows to take into account different subspaces in each way.

The unitary column vectors in **K**, **B** and **C** are then used to obtain the corresponding elements of the three-way LC model using the proper weights in **G**, for instance:

$$\boldsymbol{\kappa}_{i1} = \boldsymbol{K}_{i1}g_{111} \quad i = 1, \dots, I \tag{5}$$

As mentioned before, the advantage of introducing a third way in the analysis of the traditional Years per Age-group mortality data is much more clear if we can hypothesize that the decomposition along the third way brings information about the countries or across some two-way interactions. Now we give some further remarks.
In this context, we can distinguish three cases:

i) the different countries show homogeneous mortality patterns;

ii) the different countries have idiosyncratic patterns;
iii) there exist subgroups of countries with homogeneous trends within the same subgroup while showing different trends among the subgroups.

In the first case, since the mortality data are coherent, it is possible to aggregate the different mortality experiences. On the other hand, a factorial decomposition will provide a single component that will explain much of the inherent variability.
In the second case, any data aggregation is awkward, and any solution could lead to unreliable results. In this case, a factorial decomposition will give a poor synthesis on the first component.
The third case is more interesting from our point of view. In this case, we may argue that a unique synthesis is not reliable, but several syntheses are possible and they should be explored in this framework.
Furthermore, the presence of different patterns should be indicated by a significant source of variation across the Countries or Age-groups or by considering interactions across Countries and some age-specific element. Indeed, the first step of a mortality data investigation should be addressed by exploring the different sources of variation and highlighting the significant ones. The basic condition to analyze a mortality data structure is the evidence of trend patterns along the Years and observing usual Age-groups effects. Moreover, in our analysis we deal with the main effect and any two-way interactions across the three modes.
The main effects, two-way and the three-way interaction can be easily explored in the context of a three-way ANOVA model. However, classical conditions for proper statistical inference do not hold in this framework and just the magnitude of the effects should be presented. The main effects should provide evidence of large differences in the average levels of mortality trends either across Years, or Age-groups or Countries.
The Age-group and Countries interaction could allow us to explore the relative patterns of mortality rates for each specific Country in any Age-group and, conversely, which Age-group is affected by a relative high (low) mortality rate for each Country. The Years and Age-group two-way interaction represents the trends of the various Age-groups across the Years. Plotting Year versus Age-group gives the classical representation before extrapolating the pooled κ_i parameters (see Figure 2 and the Appendix).
The Years and Countries interaction shows how the trends vary with respect to each specific country, and so we can explore which years have relative high (low) mortality rates for any country, and which country shows a relative high (low) impact in each specific year. The presence of an important contribution of the three-way interaction is at the basis for a meaningful three-mode component analysis.

4. The case study

The above methodology is applied to the mortality data of 18 countries for male populations. Similar analysis could be provided for female data and for the aggregate population too. The third way comprises the following 18 European countries: *Austria, Sweden, UK, France, Belgium, Netherlands, Switzerland, Portugal, Italy, Norway, Spain, Finland, Luxembourg, Ireland, West Germany, East Germany, Czech Republic, Denmark*. The data were extracted from the *Human Mortality Database* (HMD 2017). The database offers information on death counts, exposure to risk and death rates for national populations (countries or areas); at present, the database contains detailed population and mortality data for 39 countries or areas. As regards our choice to include data from West and East Germany separately, it can be justified because of the differences in the collection of population statistics between the two parts of Germany before 1990 resulted from the different definitions used for live births and resident population. In the German Democratic Republic (GDR), the term “live birth” was used when the new-born exhibited at least two signs of life instead of one as in the Federal Republic of Germany (FRG) (for major details we refer the interested reader to the section on *Birth Count Data*, HMD). Also, in the GDR, foreigners who had arrived in the country less than six months before were not included in the “resident population” whereas in the FRG, all residents were included. Moreover, we decide to exclude countries like *Estonia*, *Latvia* and *Lithuania* because of the quality of population estimates for 1959-1979 that is lower than in later years. From the HMD database there is a recommendation to use these data with caution (for details, please see the "Data Quality Issues" section of the Background and Documentation file, HMD). A similar motivation guides us to exclude the Greek data. There are no data available for countries as *Romania* or *ex-Yugoslavia*.

As regards the time range, we use data from 1960 until 2006. It is well known that in this period there are some troubles: to avoid problems with zero death counts, we used mortality rates ($m_{x,t}$) by 5-year Age-groups and 1-year time-period. More specifically data were downloaded in age groups ranging from 0, 1-4, 5-9, 10-14, up to 95-99. To sum up, the data array is constituted by 18 countries, 21 age-groups and 47 years.
The general data patterns are quite similar for all 18 countries. This is shown in Figure 2 for the UK dataset, as a prototype. A noticeable feature of the pattern is the presence of different mortality levels across Age-groups. Specifically, we highlight the atypical mortality levels for the first year of life, specifically age “0”.

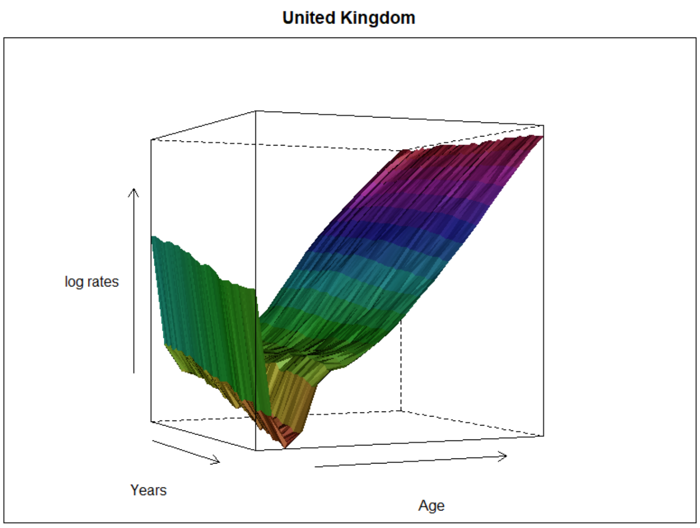

Figure 2. Log-mortality data of the United Kingdom across 47 years and 21 age-groups. The age 0 has a mortality level similar to the intermediate age-groups.

From Figure 2, it appears that the first Age-group “0” experiences a peculiar pattern which is not consistent with the contiguous Age-groups, i.e. 1-4, 5-9, etc. Conversely,

it seems to be coherent with mortality levels of middle Age-groups (50-54, 55-59, etc.). To deal with a more homogenous data structure, we decide to remove age “0” from the dataset. Thus, the three-way data array becomes of 47 Years, 20 Age-groups and 18 Countries.
In the Appendix, we display the 18 mortality data patterns, where Countries are in alphabetical order. Here, it should be pointed out that Countries show quite homogeneous mortality patterns and we may look for a consistent synthesis.
In order to assess the source of variability in the three modes, we perform a 3-way Analysis of Variance (ANOVA) with fixed effects. This kind of analysis allows us to determine the most important sources of variability in the three-way raw data structure given, in our application, by the centered log mortality rates.

The results are shown in Table 1, where it is evident that the main effect of Age-group is the one with the larger impact, accounting for the 98% of the total variability.

Total ssq	=	90958.943	%
SS_Years	=	804.657	(0.88)
SS_Age	**=**	**89164.565**	**(98.03)**
SS_Country	=	251.415	(0.28)
SS_YxA	=	329.713	(0.36)
SS_YxC	=	43.759	(0.05)
SS_AxC	=	191.889	(0.21)
SS_YxAxC	=	172.944	(0.19)

Table 1. ANOVA Results

Coherently with our idea to explore the main source of variability, we further explore the dataset by running a Hierarchical Cluster analysis (Ward’s Method), which should give useful insights on how to separate data along Age-groups. To this aim, firstly the three-way array is unfolded producing a two-way array of size (Years x Countries) x Age-groups, that is 846 x 20. Then, we compute a distance matrix holding the (20 x 20) Euclidean distances between any pairs of Age-groups. Figure 3 displays the resulting dendrogram, where four classes of homogeneous Age-groups can be distinguished: *Infant* (age groups from 1-4 to 10-14 years old), *Young* (Age-groups from 15-19 to 45-49), *Adult* (from 50-54 to 75-79) and *Old* (from 80-84 to 95-99).

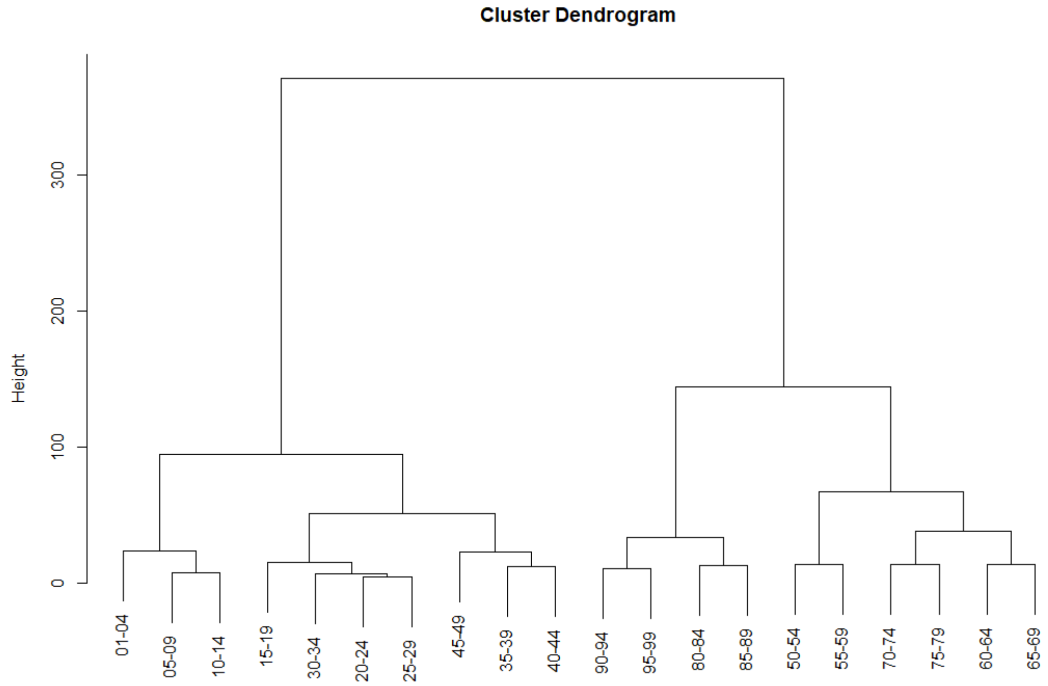

Figure 3. Homogeneous Age-groups according to the general level of mortality across Countries

In other words, it looks like in the raw data there were four possible different hidden sub-models and these sub-populations have been identified by a "data driven" approach.

Since we know that similar age patterns across populations are also an important requirement for using three-way analysis to forecast mortality (Boucher et al (2018)), we consider worthwhile to pursue the analysis by examining the four sub-groups with similar age patterns in order to obtain coherent mortality forecasting for those age groups.

Thus, we split the dataset into four homogeneous sub-groups labelled as: *Infant*, *Young*, *Old* and *Adult* Age-groups. We run the *Tucker3* analysis, for each of the four sub-groups and extract the first components corresponding to the $\kappa_{t,s}$ where s = {*Infant*: [1-14]; *Young*: [15–49]; *Adult*: [50–79]; *Old*: [80–99]}, plotted in Figure 4. The goodness of fit results from the four analyses are given in Table 2 and show reliable results for the chosen number of factors. The choices of suitable number of extracted factors is addressed by internal optimal criterion provided by most of the three-way software and therefore it is not discussed here.

	Infant	***Young***	***Adult***	***Old***
Extracted factors	$2 \times 2 \times 2$	$2 \times 2 \times 2$	$2 \times 2 \times 2$	$2 \times 2 \times 2$
SSE	77.578	109,662	14.545	22.875
R-Squared	0.9087	0,7885	0.9433	0.8650
Generalized Cross Validation (prediction error)	0.0341	0.0194	0.0047	0.0048

Table 2 – Summary of Goodness of Fit Information for the four estimated Tucker3 models.

Figure 4 clearly shows the peculiar patterns in the four sub-groups. Specifically, we notice a rapid decreasing of mortality level for *Infant* Age-group, along with the considered years range. In latest years, the *Infant* [0-14] and the *Young* Age-group [15-49] seem to converge together towards a lower mortality level. The *Adult* [50-79] and the *Old* [80-99] Age-groups seem stay close along all the years and they converge together to a higher mortality level. In other words, the *Adult* [50-79] and the *Old* [80-99] Age-groups seem stay close along all the time period, then they present a bifurcation around the year 2002 and after that, *Adult* and *Old* converge together to a higher mortality level with respect to *Infant* and *Young* people.

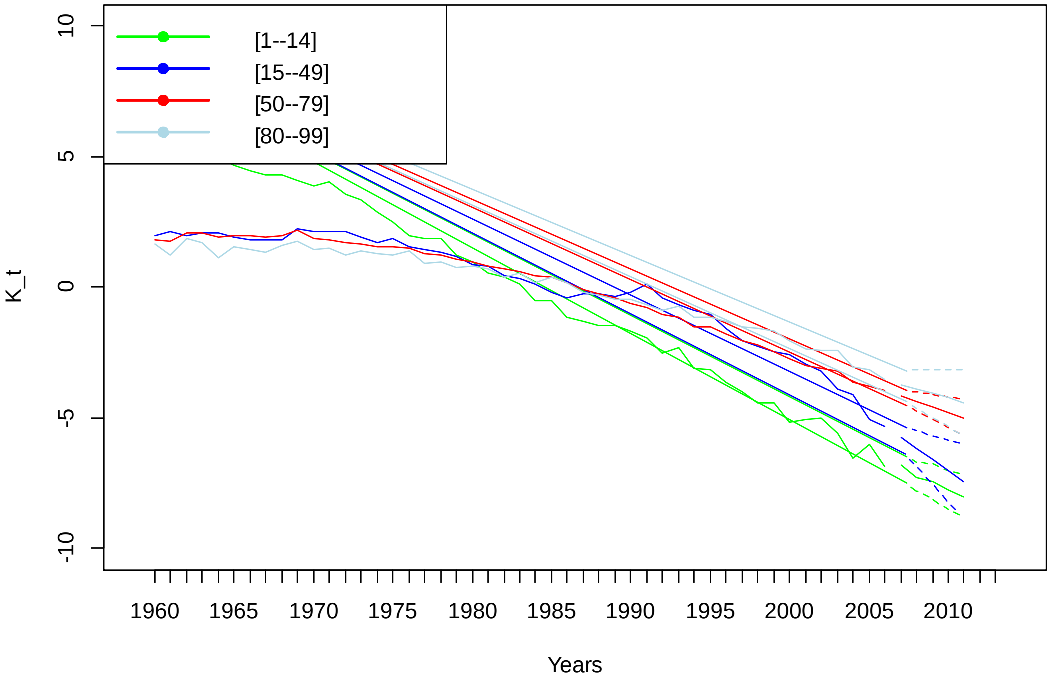

Figure 4. The four κ_t for the respective age-groups, their projections 5 years ahead and confidence regions (alfa=0.05).

These divergences emerging from the three-way analysis applied to each sub-group, coming from the intensity of the general mortality process experienced by each age group, would be generally disregarded if the different sources of variability would be not taken into account. This appears to us to be an important result enabling us to obtain a more reliable and coherent time series κ_t, which allows to project mortality rates (and hence an applicable life table) for each specific age group.

To complete the analysis, we derived five-years ahead forecasts and the corresponding 95% confidence regions (see Figure 4). Definitively, we can highlight that *Infant* and *Young* show overlapping regions, as well as *Adult* and *Old* age groups do. However, *Infant* and *Young* trends are well separated from the *Adult* and *Old* ones. We have seen that the numerical analysis seems to confirm the Age-groups clustering depicted in the Dendrogram in Figure 3. In order to forecast the deriving time series, we estimate four sub-models and highlighting that they underlie different stochastic processes. The estimated ARIMA models for the four trends showed in Figure 4 are depicted in Table 3.

	Infant	*Young*	*Adult*	*Old*
ARIMA specification	1,1,0	0,2,1	0,2,1	0,2,1
AR: ϕ	-0.5186 (0.1309)			
Drift: d	-0.2696 (0.0249)			
MA: θ		-0.7865 (0.0944)	-0.8261 (0.0605)	-0.9338 (0.0572)
Sigma2; logLikelihood	0.067; -2.42	0.050; 3.5	0.014; 32.32	0.064; -3.19
AIC; BIC	10.83; 16.32	-2.99; 0.62	-60.64; -57.02	10.38; 13.99

Tab. 3 – ARIMA specifications and statistical modelling of the four age-groups.

The results in Table 3 display the ARIMA model specifications, the estimated parameters and model fitting statistics. The first thing to notice is that the Infant Age-group has an idiosyncratic model specification in the class of AR model with drift; this should be due to the relatively rapid decrease of the mortality levels in the considered time period for the Infant Age-group. The other three Age-groups, despite showing a convenient representation in the same class of stochastic process (MA model with a quadratic integration to obtain suitable stationarity), show that the MA coefficient θ tend to |1| when age increases, ranging from -0.7865 (Young) to -0.9338 (Old). From a statistical point of view, this implies that when the age increases, modeling mortality data for projection purposes is meaningless in this class of models.

5. Conclusions

In Section 4, we consider 18 countries by 5-year Age-groups, for the time period 1960–2006. We construct the males Three-way data array that initially consists of 18 Countries, 21 Age groups and 47 Years. After noticing atypical mortality levels for the first age-group "0", we remove this age from the data set in order to obtain a more homogeneous data structure. Coherently with our main aim, we explore the main sources of variability in the three-way data structure given by the centered log mortality rates.

The application of the Tucker3 model to the case study provides an interesting visualization of the relationships between the components of the mortality data along years and Age-groups. Moreover, the analysis allows us to discuss the coherence of a composite trend across different countries and their relative homogeneity patterns.

Following a "data driven" approach, we find out that in the raw data four possible different sub-models are hidden. After identifying these sub-populations, we extract coherent mortality forecast for those age homogeneous sub-groups.

The proposed procedure presents many potential advantages. It allows to empirically verify the heterogeneity in the raw data structure and to analyze, with a three-way ANOVA, the most important sources of heterogeneity within sup-populations. In addition, combining exploratory factorial techniques and clustering methods, we are able to apply the three-way model and so running specific analysis with respect to each coherent class of Age-groups. This procedure has the advantage of providing synthetic κ_t for all the considered countries, while being specific for a particular age homogeneous sub-groups. Investigates with various countries, show that the mortality

dynamics are better described when the mortality experience of connected population is considered than modeling them independently.
In this paper we only deal with male data, however, as a further investigation, we plan to introduce a fourth way in the analysis, by assessing the gender effect. This 4-way analysis could provide useful insight in comparing forecast along Years, across Age-groups, Countries, and Gender.

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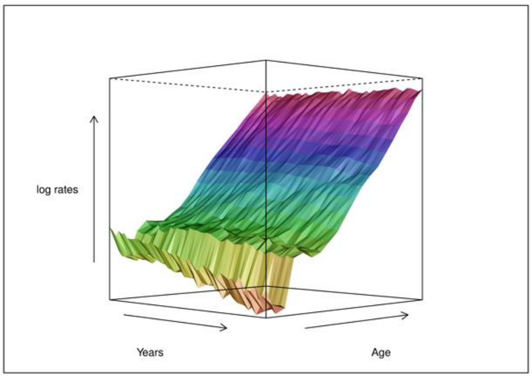
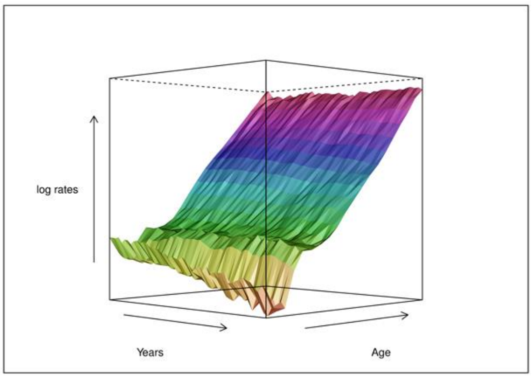
Appendix. The 18 log-mortality patterns for every countries.
Axes show Years and Age-groups. The vertical axis is the log-mortality rate.

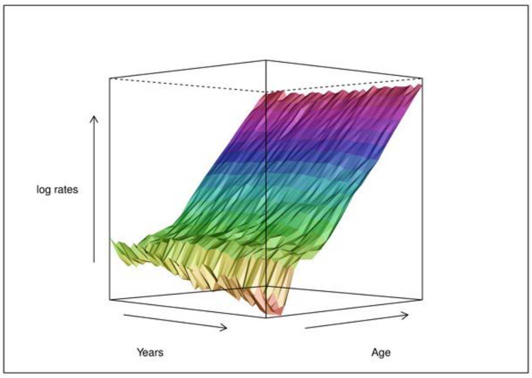
Austria



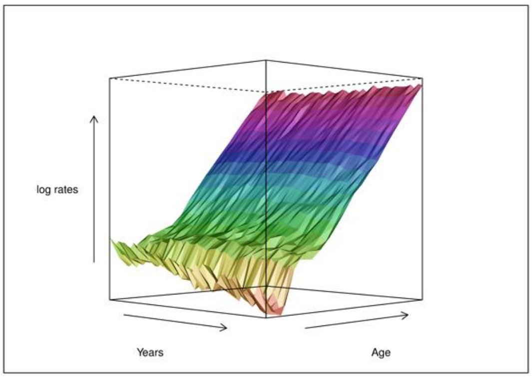
Belgium



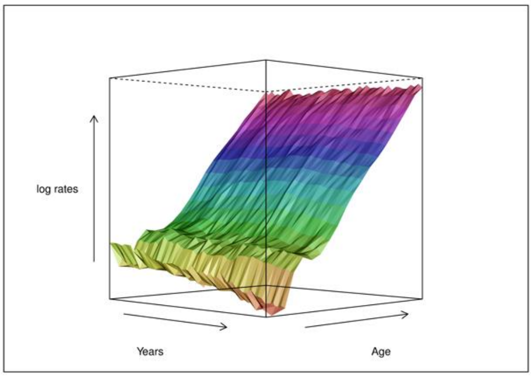
Czech Republic



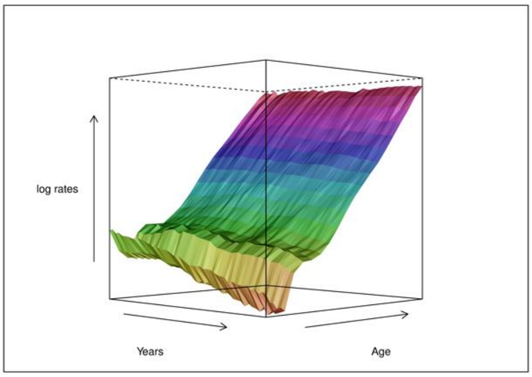
Denmark

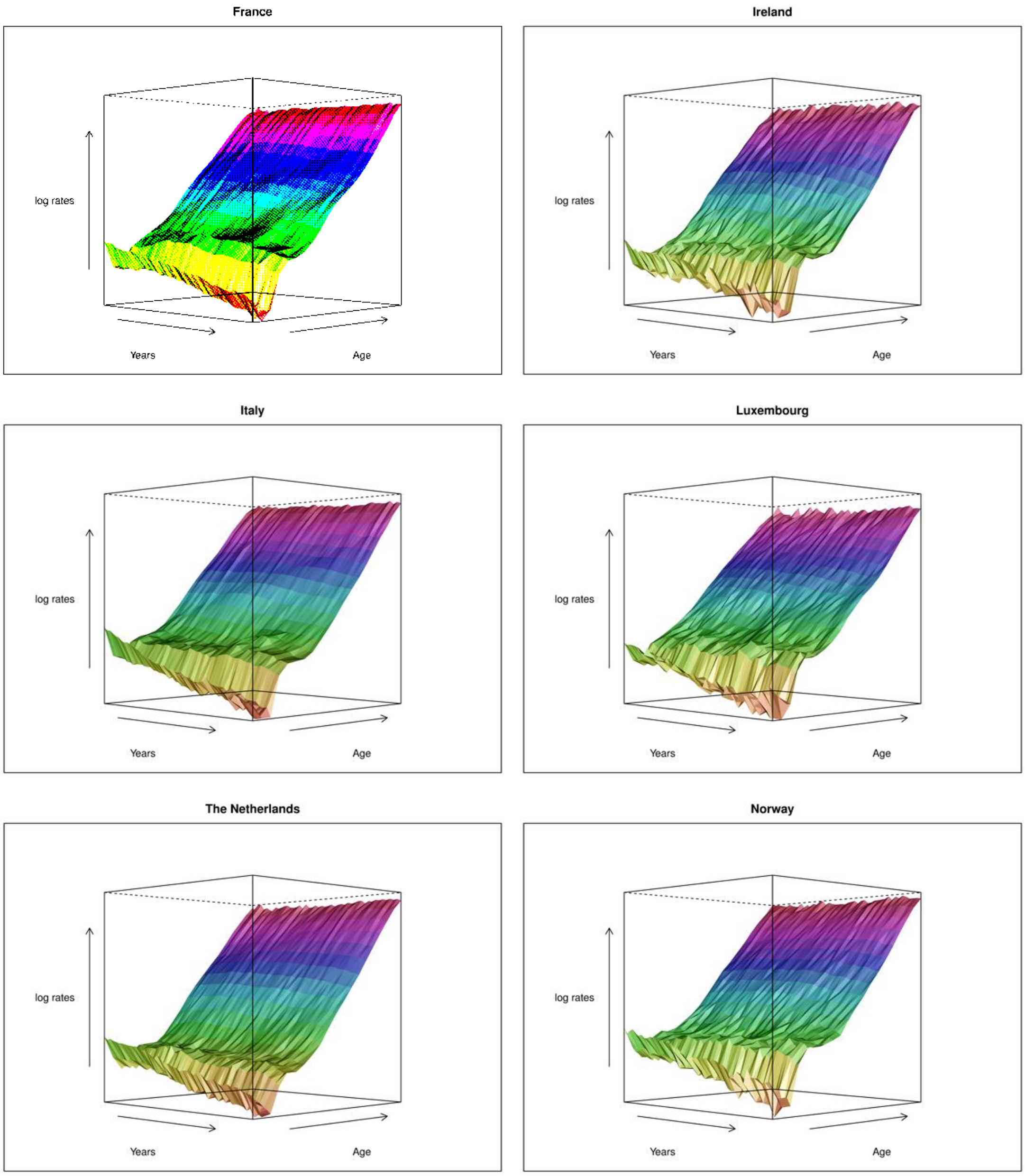


East Germany



Finland




France
log rates
Years
Age
Ireland
log rates
Years
Age
Italy
log rates
Years
Age
Luxembourg
log rates
Years
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The Netherlands
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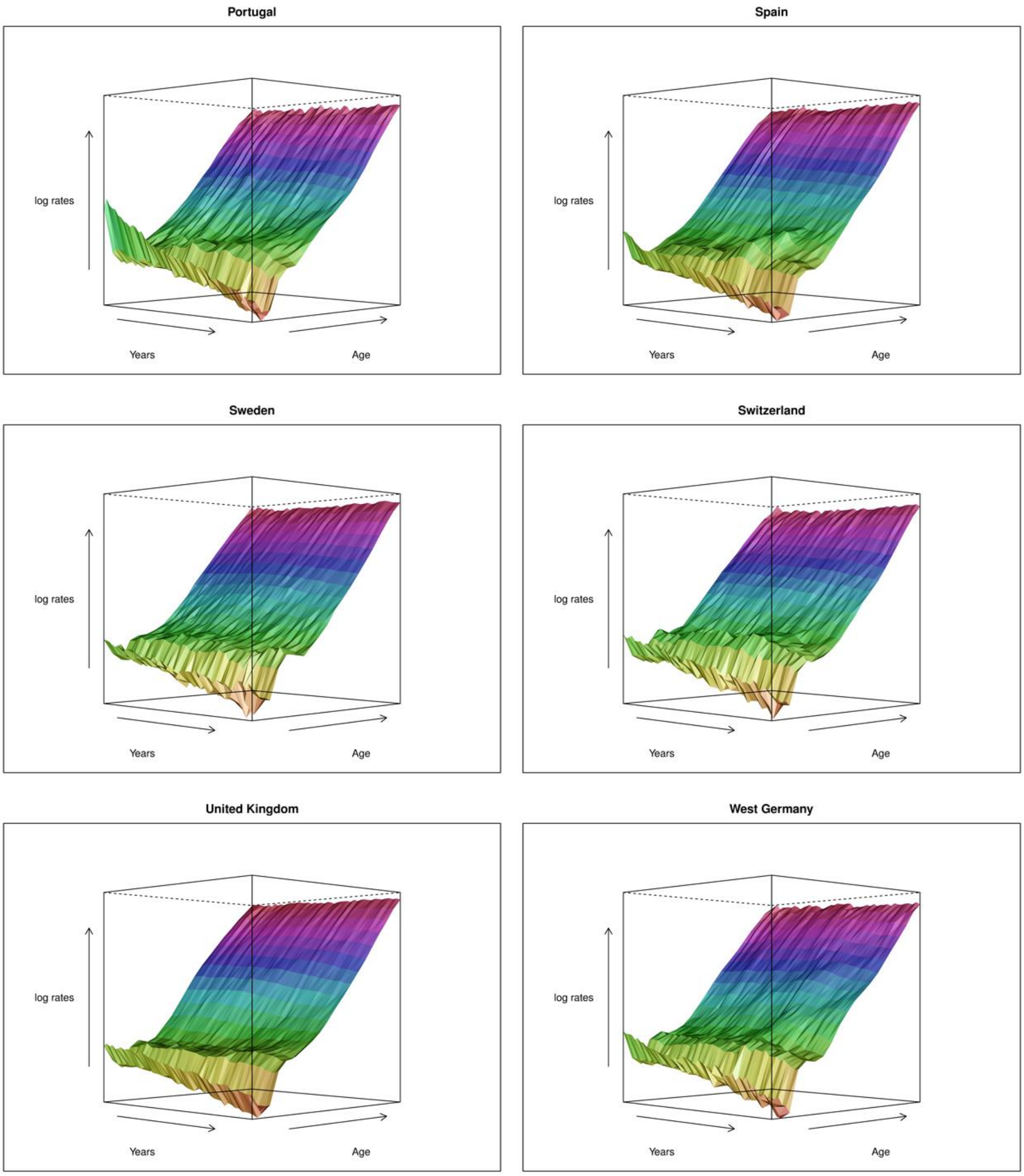
Norway
log rates
Years
Age


Portugal
Spain
Sweden
Switzerland
United Kingdom
West Germany
log rates
Years
Age